# Chapter 10

## (10) Inequalities

# **Table 10-1:** Notations for inequalities explained

| Symbol | Name | What it does / shows | Example |
|---|---|---|---|
| $<$ | Less than | ▪ The left-hand side expression (or number) is less than the right-hand side expression (or number).<br>▪ We use to represent it on a number line. Notice the direction of the arrow and the hollow circle or unfilled circular part. | ▪ $2 < 3$<br>▪ $5x - 2 < 0$ |
| $>$ | Greater than | ▪ The left-hand side expression (or number) is greater than the right-hand side expression.<br>▪ We use to represent it on a number line. | ▪ $10 > 3$<br>▪ $b - 1 > \frac{1}{4}$ |
| $\leq$ | Less than or equal to | ▪ The left-hand side expression (or number) is less than OR equal to the right-hand side expression (or number).<br>▪ It is possible to think about this expression as a combination of $<$ and $=$. Alternatively, it is like when we use 'inclusive'. For example, our holiday is from Monday to Saturday inclusive, which means Monday and Saturday are included in the holiday.<br>▪ We use to represent it on a number line. Notice the solid circle or filled circular part. It stands for the $=$ in the overall symbol. | ▪ $y + 1 \leq y^2 - 4y$ |
| $\geq$ | Greater than or equal to | ▪ The left-hand side expression (or number) is greater than OR equal to the right-hand side expression (or number).<br>▪ It is possible to think about this expression as a combination of $>$ and $=$.<br>▪ We use to represent it on a number line. | ▪ $ab - a \leq a^2 + 4$ |

## **Table 10-2:** Complementary inequalities illustrated

| | Inequality | Members |
|---|---|---|
| i) | $x < -2$ | $-5, -4, -3$ |
| | $x \geq -2$ | $-2, -1, 0, 1, 2, 3, 4, 5$ |
| ii) | $x > -2$ | $-1, 0, 1, 2, 3, 4, 5$ |
| | $x \leq -2$ | $-5, -4, -3, -2,$ |

## **Table 10-3:** Rules of addition and subtraction illustrated

| Inequality | Addition | Subtraction |
|---|---|---|
| $\boldsymbol{a > b}$ | $a + c > b + c$<br>For example, if $13 > 5$ then $13 + 4 > 5 + 4$ is also true. That is $17 > 9$. | $a - c > b - c$<br>For example, if $13 > 5$ then $13 - 4 > 5 - 4$ is also true. That is $9 > 1$. |
| $\boldsymbol{a < b}$ | $a + c < b + c$<br>For example, if $7 < 21$ then $7 + 4 < 21 + 4$ is also true. That is $11 < 25$. | $a - c < b - c$<br>For example, if $7 < 21$ then $7 - 4 < 21 - 4$ is also true. That is $3 < 17$. |
| $\boldsymbol{a \geq b}$ | Given that $a \geq b$, it follows that<br>$a + c \geq b + c$<br>is also valid. | Given that $a \geq b$, it follows that<br>$a - c \geq b - c$<br>is also valid. |
| $\boldsymbol{a \leq b}$ | Given that $a \leq b$, it follows that<br>$a + c \leq b + c$<br>is also valid. | Given that $a \leq b$, it follows that<br>$a - c \leq b - c$<br>is also valid. |

## **Table 10-4:** Rules of multiplication and division illustrated for positive factor

| Inequality | Multiplication | Division |
|---|---|---|
| $a < b$ | $$ac < bc$$ For example, if $1 < 2$ then $1 \times 3 < 2 \times 3$ is also true. That is $3 < 6$. | $$\frac{a}{c} < \frac{b}{c}$$ For example, if $4.2 < 6$ then $\frac{4.2}{3} < \frac{6}{3}$ is also true. That is $1.4 < 2$. |
| $a > b$ | $$ac > bc$$ For example, if $5 > 2$ then $5 \times 3 > 2 \times 3$ is also true. That is $15 > 6$. | $$\frac{a}{c} > \frac{b}{c}$$ For example, if $27 > 21$ then $\frac{27}{3} > \frac{21}{3}$ is also true. That is $9 > 7$. |
| $a + b \leq 1$ | Given that $a + b \leq 1$, it follows that $$(a + b)c \leq c$$ is also valid. | Given that $a + b \leq 1$, it follows that $$\frac{a + b}{c} \leq \frac{1}{c}$$ is also valid. |
| $a + b \geq p$ | Given that $a + b \geq p$, it follows that $$(a + b)c \geq pc$$ is also valid for $c > 0$. | Given that $a + b \geq p$, it follows that $$\frac{(a + b)}{c} \geq \frac{p}{c}$$ is also valid for $c > 0$. |

**Table 10-5:** Rules of multiplication and division illustrated for negative factor

| Inequality | Multiplication | Division |
|---|---|---|
| $\boldsymbol{a > b}$ | $ac < bc$<br>For example, if $7 > -4$ then $7 \times -2.5 < -4 \times -2.5$ is also true. That is $-17.5 < 10$. | $\frac{a}{c} < \frac{b}{c}$<br>For example, if $9 > 4$ then $\frac{9}{-2.5} < \frac{4}{-2.5}$ is also true. That is $-3.6 > -1.6$. |
| $\boldsymbol{a < b}$ | $ac > bc$<br>For example, if $-4 < 1$ then $-4 \times -2.5 > 1 \times -2.5$ is also true. That is $10 > -2.5$. | $\frac{a}{c} < \frac{b}{c}$<br>For example, if $15 < 25$ then $\frac{15}{-2.5} > \frac{25}{-2.5}$ is also true. That is $-6 > -10$. |
| $\boldsymbol{a - b \leq 1}$ | Given that $a - b \leq 1$, it follows that<br>$(a - b)c \geq c$<br>is also valid for $c < 0$. | Given that $a - b \leq 1$, it follows that<br>$\frac{a - b}{c} \geq \frac{1}{c}$<br>is also valid for $c < 0$. |
| $\boldsymbol{a - b \geq p}$ | Given that $a - b \geq p$, it follows that<br>$(a - b)c \leq pc$<br>is also valid for $c < 0$. | Given that $a - b \geq p$, it follows that<br>$\frac{(a - b)}{c} \leq \frac{p}{c}$<br>is also valid for $c < 0$. |

**Table 10-6:** Alternative method to reversing inequality sign when multiplying and dividing with a negative factor illustrated

| Using the rule | Alternative method |
|---|---|
| Using the rule, we will divide both sides by $-2$ and flip the direction of the inequality as: $$-2x < 6$$ $$-\frac{2x}{-2} > \frac{6}{-2}$$ $$x > -3$$ | Alternatively, we can move $-2x$ to the right and 6 to the left and divide by 2 as: $$-6 < 2x$$ $$-\frac{6}{2} < \frac{2x}{2}$$ $$-3 < x$$ |

## **Table 10-7:** Rules of power illustrated

| Inequality | Positive power (for $m = 2$) | Negative power (for $m = -2$) |
|---|---|---|
| $a < b$ | $a^m < b^m$<br>For example, if $6 < 7$ then $6^2 < 7^2$ is also true. That is $36 < 49$. | $a^m > b^m$<br>For example, if $10 < 15$ then $10^{-2} > 15^{-2}$ is also true. That is $\frac{1}{10^2} > \frac{1}{15^2}$. |
| $a > b$ | $a^m > b^m$<br>For example, if $10 > 5$ then $10^2 > 5^2$ is also true. That is $100 > 25$. | $a^m > b^m$<br>For example, if $8 > 3$ then $8^{-2} < 3^{-2}$ is also true. That is $\frac{1}{8^2} < \frac{1}{3^2}$. |
| $a + b \leq c$ | Given that $a + b \leq c$, it follows that<br>$(a+b)^m \leq c^m$<br>is also valid. | Given that $a + b \leq c$, it follows that<br>$(a+b)^{-m} \geq c^{-m}$<br>is also valid. |
| $a + b \geq c + d$ | Given that $a + b \geq c + d$, it follows that<br>$(a+b)^m \geq (c+d)^m$<br>is also valid. | Given that $a + b \geq c + d$, it follows that<br>$(a+b)^{-m} \leq (c+d)^{-m}$<br>is also valid. |

## **Table 10-8:** Rules of inversion illustrated

| Inequality | Inversion |
|---|---|
| $\boldsymbol{a > b}$ | $b < a$ |
| $\boldsymbol{m + n < 0}$ | $0 > m + n$ |
| $\boldsymbol{s + c \leq t + d}$ | $t + d \geq s + c$ |
| $\boldsymbol{u \geq v}$ | $v \leq u$ |

## **Table 10-9:** Rules of inversion illustrated

| Inequality | Solution | Example |
|---|---|---|
| $x^2 < a$ | $-\sqrt{a} < x < \sqrt{a}$ | $x^2 < 5$ implies $-\sqrt{5} < x < \sqrt{5}$ |
| $x^2 > a$ | $x > \sqrt{a}$ and $x < -\sqrt{a}$ | $x^2 > 4$ implies $x > \sqrt{4}$ and $x < -\sqrt{4}$<br>This can be further simplified to<br>$x > 2$ and $x < -2$ |
| $x^2 \leq b$ | $-\sqrt{b} \leq x \leq \sqrt{b}$ | $(x-3)^2 \leq 10$ implies $-10 \leq (x-3) \leq 10$<br>This can be further simplified to<br>$-10 \leq x-3 \leq 10$<br>$-7 \leq x \leq 13$ |
| $x^2 \geq b$ | $x \geq \sqrt{b}$ and $x \leq -\sqrt{b}$ | $(2x-5)^2 - 9 \geq 0$ this can be re-written as<br>$(2x-5)^2 \geq 9$<br>This implies<br>$2x-5 \geq \sqrt{9}$ and $2x-5 \leq -\sqrt{9}$<br>This can be further simplified to<br>$2x-5 \geq 3$ and $2x-5 \leq -3$<br>$x \geq 4$ and $x \leq 1$ |

# **Table 10-10:** Author's method illustrated

| Option | Solution | Condition | Example |
|---|---|---|---|
| **1)** | $x < \alpha$ and $x > \beta$ | ▪ $a > 0$ and $y > 0$<br>▪ $a < 0$ and $y < 0$ | ▪ $x^2 - 2x - 3 > 0$<br>▪ $5 - 3x - 2x^2 < 0$ |
| **2)** | $\alpha < x < \beta$ | ▪ $a > 0$ and $y < 0$<br>▪ $a < 0$ and $y > 0$ | ▪ $x^2 - 2x - 3 < 0$<br>▪ $5 - 3x - 2x^2 > 0$ |

**Table 10-11:** Solving quadratic inequalities using intervals illustrated – Part I

| | | $x < \alpha$ | $\alpha < x < \beta$ | $x > \beta$ |
|---|---|---|---|---|
| R1 | $(x - \alpha)$ | | | |
| R2 | $(x - \beta)$ | | | |
| R3 | $(x - \alpha)(x - \beta)$ | | | |

**Table 10-12:** Solving quadratic inequalities using intervals illustrated – Part II

| | | $x < \alpha$ | $\alpha < x < \beta$ | $x > \beta$ |
|---|---|---|---|---|
| **R1** | $(x - \alpha)$ | $-$ | $-$ | $+$ |
| **R2** | $(x - \beta)$ | $-$ | $+$ | $+$ |
| **R3** | $(x - \alpha)(x - \beta)$ | | | |

## **Table 10-13:** Solving quadratic inequalities using intervals illustrated – Part III

| | | $x < \alpha$ | $\alpha < x < \beta$ | $x > \beta$ |
|---|---|---|---|---|
| **R1** | $(x-\alpha)$ | $-$ | $-$ | $+$ |
| **R2** | $(x-\beta)$ | $-$ | $+$ | $+$ |
| **R3** | $(x-\alpha)(x-\beta)$ | $+$ | $-$ | $+$ |

**Table 10-14:** Solution to Example 11(a) – Part I

| | $x < -3$ | $-3 < x < 5$ | $x > 5$ |
|---|---|---|---|
| $(x+3)$ | | | |
| $(x-5)$ | | | |
| $(x+3)(x-5)$ | | | |

## **Table 10-15:** Solution to Example 11(a) – Part II

| | $x < -3$ | $-3 < x < 5$ | $x > 5$ |
|---|---|---|---|
| $(x+3)$ | $-$ | $+$ | $+$ |
| $(x-5)$ | $-$ | $-$ | $+$ |
| $(x+3)(x-5)$ | $+$ | $-$ | $+$ |

## **Table 10-16:** Solution to Example 11(b) – Part I

| | $x \le \frac{1}{2}$ | $\frac{1}{2} \le x \le \frac{5}{3}$ | $x \ge \frac{5}{3}$ |
|---|---|---|---|
| $(2x-1)$ | | | |
| $(3x-5)$ | | | |
| $(2x-1)(3x-5)$ | | | |

# **Table 10-17:** Solution to Example 11(b) – Part II

| | $x \le \frac{1}{2}$ | $\frac{1}{2} \le x \le \frac{5}{3}$ | $x \ge \frac{5}{3}$ |
|---|---|---|---|
| $(2x-1)$ | $-$ | $+$ | $+$ |
| $(3x-5)$ | $-$ | $-$ | $+$ |
| $(2x-1)(3x-5)$ | $+$ | $-$ | $+$ |

## **Table 10-18:** Rules of inversion illustrated

| Inequality | Solution | Example |
|---|---|---|
| $\lvert x \rvert < a$ | $-a < x < a$ | $\lvert x \rvert < 3$ implies $-3 < x < 3$ |
| $\lvert x \rvert > a$ | $x > a$ and $x < -a$ | $\lvert x \rvert > 4$ implies $x > 4$ and $x < -4$ |
| $\lvert x \rvert \leq b$ | $-b \leq x \leq b$ | $\lvert x \rvert \leq 5$ implies $-5 \leq x \leq 5$ |
| $\lvert x \rvert \geq b$ | $x \geq b$ and $x \leq -b$ | $\lvert x \rvert \geq 6$ implies $x \geq 6$ and $x \leq -6$ |

# Thank You